

# Overpartitions with bounded part differences

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**Abstract.** We generalize recent results of Breuer and Kronholm, and Chern on partitions and overpartitions with bounded differences between largest and smallest parts. We prove the generalization analytically and combinatorially.

**Keywords.** Partition, overpartition, bounded difference between largest and smallest parts, combinatorial proof.

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## 1. Introduction

A *partition* of a positive integer  $n$  is a non-increasing sequence of positive integers whose sum is  $n$ . For example, there are three partitions of 3:  $3$ ,  $2 + 1$ , and  $1 + 1 + 1$ . Recently, Andrews *et al.* [2] obtained the generating function for partitions with the difference between largest and smallest parts being a given positive integer  $t$ . Motivated by the work of Andrews *et al.*, Breuer and Kronholm [3] studied partitions in which the difference between largest and smallest parts is at most  $t$ , and they showed that the generating function for such partitions is

$$\sum_{n \geq 1} p_t(n) q^n = \frac{1}{1 - q^t} \left( \frac{1}{(q)_t} - 1 \right), \quad (1.1)$$

where  $p_t(n)$  counts the number of partitions of  $n$  with part differences at most  $t$ .

Here and in the sequel, we use the standard  $q$ -series notation

$$(a)_n = (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k).$$

The proof of Breuer and Kronholm has geometric flavor and their main tool used in the proof is polyhedral cones. Subsequently, Chapman [4] also provided a simpler proof, which involves  $q$ -series manipulations.

An *overpartition* of  $n$  is a partition of  $n$  where the first occurrence of each distinct part may be overlined. For example, there are eight overpartitions of 3:  $3$ ,  $\overline{3}$ ,  $2 + 1$ ,  $\overline{2} + 1$ ,  $2 + \overline{1}$ ,  $\overline{2} + \overline{1}$ ,  $1 + 1 + 1$ , and  $\overline{1} + 1 + 1$ . Recently, motivated by the works of Andrews *et al.*, Breuer and Kronholm, and Chapman, the first author [5] considered an overpartition analogue with bounded differences between largest and smallest parts.

To obtain a generating function analogous to (1.1), apart from requiring the difference between largest and smallest parts being at most a given positive integer

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$t$ , he added the following restriction: if the difference between largest and smallest parts is exactly  $t$ , then no largest parts can be overlined. Let  $g_t(n)$  count the number of such overpartitions of  $n$ , then its generating function is

$$\sum_{n \geq 1} g_t(n) q^n = \frac{1}{1 - q^t} \left( \frac{(-q)_t}{(q)_t} - 1 \right). \quad (1.2)$$

The first proof in his paper uses heavy  $q$ -series manipulation, which originates from [2]. His second proof, which consists of many combinatorial ingredients such as over  $q$ -binomial coefficient introduced by Dousse and Kim [6], however, still needs some nontrivial computation, and hence is also not completely combinatorial.

The main purpose of this paper is to provide a completely combinatorial and transparent proof of (1.2). More precisely, we prove the following refined result.

**Theorem 1.1.** *For a positive integer  $t$ , let  $g_t(m, n)$  count the number of overpartitions of  $n$  in which there are exactly  $m$  overlined parts, the difference between largest and smallest parts is at most  $t$ , and if the difference between largest and smallest parts is exactly  $t$ , then no largest parts are overlined. Then*

$$\sum_{n \geq 1} \sum_{m \geq 0} g_t(m, n) z^m q^n = \frac{1}{1 - q^t} \left( \frac{(-zq)_t}{(q)_t} - 1 \right). \quad (1.3)$$

We remark that (1.1) and (1.2) follow immediately by taking  $z \rightarrow 0$  and  $z \rightarrow 1$  respectively.

**1.1. Notation and terminologies.** Throughout this paper,  $\mathbb{Z}_{\geq 0}$  and  $\mathbb{Z}_{> 0}$  denote the set of nonnegative integers and positive integers, respectively. Given a partition or an overpartition  $\lambda$  of  $n$ , let  $\ell(\lambda)$  be the number of parts of  $\lambda$  and  $|\lambda| = n$  be the sum of the parts of  $\lambda$ . When  $\lambda$  is an overpartition, we use  $o(\lambda)$  to count the number of overlined parts in  $\lambda$ . We write parts in weakly decreasing order.

For a positive integer  $t$ , we denote by  $\overline{\mathcal{P}}_t$  the set of (nonempty) overpartitions with parts less than or equal to  $t$  and no parts equal to  $t$  overlined, and by  $\overline{\mathcal{G}}_t$  the set of (nonempty) overpartitions with differences between largest and smallest parts at most  $t$  and no largest parts overlined when the difference between largest and smallest parts is exactly  $t$ . Also,  $\overline{\mathcal{B}}_t$  denotes the set of bipartitions where the first subpartition, which can be an empty partition, consists of only parts equal to  $t$ , all not overlined, and the second subpartition is a nonempty overpartition with the largest part at most  $t$ .

The rest of this paper is organized as follows. In Section 2.1, we first construct a weight preserving map  $\phi$  from  $\overline{\mathcal{G}}_t$  to  $\overline{\mathcal{P}}_t$ . In Section 2.2, we then construct another weight preserving map  $\psi$  from  $\overline{\mathcal{P}}_t$  to  $\overline{\mathcal{B}}_t$ . Finally, by combining these two maps, we will deduce that  $\overline{\mathcal{G}}_t$  and  $\overline{\mathcal{B}}_t$  have the same generating functions:

$$\sum_{\pi \in \overline{\mathcal{G}}_t} z^{o(\pi)} q^{|\pi|} = \sum_{\beta \in \overline{\mathcal{B}}_t} z^{o(\beta)} q^{|\beta|},$$

which is indeed equivalent to Theorem 1.1. In Section 3, a  $q$ -series proof of Theorem 1.1 will be given.

## 2. A combinatorial approach

**2.1. Partition Sets  $\overline{\mathcal{G}}_t$  and  $\overline{\mathcal{P}}_t$ .** For an overpartition  $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$  in  $\overline{\mathcal{G}}_t$ , let  $s(\pi) = \lfloor \pi_\ell / t \rfloor$ , where  $\lfloor a \rfloor$  denotes the largest integer not exceeding  $a$ , and let

$k(\pi)$  be a positive integer  $k$  such that  $\pi_k \geq (s(\pi) + 1)t$  and  $\pi_{k+1} < (s(\pi) + 1)t$ . If there is no such  $k$ , then we let  $k(\pi) = 0$ .

We now define a map  $\phi : \overline{\mathcal{G}}_t \rightarrow \overline{\mathcal{P}}_t$  as follows. For an overpartition  $\pi \in \overline{\mathcal{G}}_t$ , let  $\ell(\pi) = \ell$ ,  $s(\pi) = s$  and  $k(\pi) = k$ . Then

$$\begin{aligned} \phi : (\pi_1, \pi_2, \dots, \pi_\ell) \\ \mapsto (\underbrace{t, t, t, \dots, t}_{s(\ell-k)+(s+1)k \text{ times}}, \pi_{k+1} - st, \dots, \pi_\ell - st, \pi_1 - (s+1)t, \dots, \pi_k - (s+1)t), \end{aligned}$$

where all the parts equal to  $t$  are not overlined, and if  $\pi_i$  is overlined, then  $\pi_i - st$  (or  $\pi_i - (s+1)t$  depending on the value of  $i$ ) is overlined. Here we note that there may be parts equal to 0 in  $\phi(\pi)$ . If there are any parts equal to 0, then we delete them so that  $\phi(\pi)$  has positive parts only.

**Theorem 2.1.**  $\phi$  is a weight preserving map from  $\overline{\mathcal{G}}_t$  to  $\overline{\mathcal{P}}_t$ .

*Proof.* Since  $\pi_1 - \pi_\ell \leq t$ ,  $s = \lfloor \pi_\ell / t \rfloor$ , and  $\pi_k \geq (s+1)t > \pi_{k+1}$ , we have

$$t > \pi_{k+1} - st \geq \dots \geq \pi_\ell - st \geq \pi_1 - (s+1)t \geq \dots \geq \pi_k - (s+1)t.$$

Thus the parts of  $\phi(\pi)$  are less than or equal to  $t$ , and if there are overlined parts, they are less than  $t$ .

We now show that no more than one part of the same size is overlined. Since  $\pi$  is an overpartition, at most one part of the same size is overlined in  $\pi$ . Hence, of  $\pi_1 - st, \dots, \pi_k - st$ , if there are overlined parts, then they must be of different sizes. For the same reason, of  $\pi_{k+1} - (s+1)t, \dots, \pi_\ell - (s+1)t$ , overlined parts must be of different sizes. Thus, if  $\pi_\ell - st > \pi_1 - (s+1)t$ , then it is clear that all the overlined parts of  $\phi(\pi)$  have different sizes.

Let us suppose that  $\pi_\ell - st = \pi_1 - (s+1)t$ . Then, we have  $\pi_1 - \pi_\ell = t$ . By the definition of  $\overline{\mathcal{G}}_t$ , we know that all the parts equal to  $\pi_1$  are not overlined. Thus, there is at most one overlined part in  $\phi(\pi)$  that is equal to  $\pi_\ell - st = \pi_1 - (s+1)t$ . Therefore,  $\phi(\pi) \in \overline{\mathcal{P}}_t$ .

We also note that the map  $\phi$  preserves the weight of  $\pi$ , that is,  $|\phi(\pi)| = |\pi|$ .  $\square$

As we see in the following example, the map  $\phi$  is not a bijection.

**Example 2.1.** Let  $t = 3$ ,  $\pi_1 = (7, \overline{4})$  and  $\pi_2 = (\overline{4}, 4, 3)$ . Then

$$\begin{aligned} s(\pi_1) = 1, \quad k(\pi_1) = 1, \quad \phi(\pi_1) = (3, 3, 3, \overline{1}, 1), \quad |\phi(\pi_1)| = |\pi_1| = 11; \\ s(\pi_2) = 1, \quad k(\pi_2) = 0, \quad \phi(\pi_2) = (3, 3, 3, \overline{1}, 1), \quad |\phi(\pi_2)| = |\pi_2| = 11. \end{aligned}$$

However,  $\phi$  is a surjection since  $\overline{\mathcal{P}}_t$  is a subset of  $\overline{\mathcal{G}}_t$  and  $\phi(\pi) = \pi$  for any  $\pi \in \overline{\mathcal{P}}_t$ . So, we will count how many pre-images each  $\mu \in \overline{\mathcal{P}}_t$  has under  $\phi$ .

Let  $\pi \in \overline{\mathcal{G}}_t$ . We describe how to recover  $\pi$  from  $\phi(\pi)$ . First, note that it is clear from the definition of  $s(\pi)$  and  $k(\pi)$  that  $\pi_i - (s(\pi) + 1)t$  and  $\pi_j - s(\pi)t$  are the remainders of  $\pi_i$  and  $\pi_j$  when divided by  $t$  for  $1 \leq i \leq k(\pi)$  and  $j > k(\pi)$ . If the remainders are equal to 0, then they are deleted in  $\phi(\pi)$ . Thus if we know the number of such deleted remainders, we can determine  $\ell(\pi)$ . Also, one of the deleted remainder may have been overlined.

We then need to find  $s(\pi)$  and  $k(\pi)$ , where  $s(\pi)$  is the quotient of the smallest part of  $\pi$  when divided by  $t$  and  $k(\pi)$  counts the number of parts whose quotients are equal to  $s(\pi) + 1$ . Therefore, once we have  $\ell(\pi)$ ,  $k(\pi)$ , and  $s(\pi)$  along with the information on existence of an overlined deleted remainder, it is clear that we

can recover  $\pi$ . Thus possible choices for  $\ell(\pi)$ ,  $k(\pi)$ , and  $s(\pi)$  with having a deleted remainder overlined or not will determine the number of pre-images under  $\phi$ .

In the following lemma, we will see the range for  $\ell(\pi)$ . For any  $\mu \in \overline{\mathcal{P}}_t$ , we use  $m(\mu) = m_t(\mu)$  to count the number of parts of  $\mu$  that equal  $t$ .

**Lemma 2.2.** *Let  $\pi$  be a nonempty overpartition in  $\overline{\mathcal{G}}_t$  and  $\mu = \phi(\pi)$  in  $\overline{\mathcal{P}}_t$ . Then we have*

- (i)  $\ell(\pi) \leq \ell(\mu)$ ;
- (ii)  $\ell(\pi) \geq \ell(\mu) - m(\mu) + \delta_{\ell(\mu), m(\mu)}$ , where  $\delta_{\ell(\mu), m(\mu)}$  is the Kronecker delta.

*Proof.* Let  $\ell(\pi) = \ell$ ,  $s(\pi) = s$ ,  $k(\pi) = k$ . We first note that, if  $s \geq 1$ , then

$$\ell(\mu) \geq s(\ell - k) + (s + 1)k = s\ell + k \geq \ell.$$

If  $s = 0$ , since  $\pi_\ell > 0$ , all of  $\pi_{k+1} - st, \dots, \pi_\ell - st$  are nonzero in  $\mu$ . Hence

$$\ell(\mu) \geq s(\ell - k) + (s + 1)k + (\ell - k) = \ell.$$

This completes the proof of (i).

Next, we prove (ii). If all of the parts of  $\mu$  are  $t$ , i.e.,  $\ell(\mu) = m(\mu)$ , then

$$\ell(\mu) - m(\mu) + \delta_{\ell(\mu), m(\mu)} = 1 \leq \ell,$$

where the last inequality follows from the fact that  $\pi$  is nonempty.

We now suppose that  $\mu$  has a part not equal to  $t$ , i.e.,  $\ell(\mu) - m(\mu) \geq 1$ . From the definition of  $\phi$ , we know that the parts of  $\mu$  not equal to  $t$  are the positive remainders of the parts of  $\pi$ , so at most  $\ell$  parts of  $\mu$  are not equal to  $t$ . Hence

$$\ell(\mu) - m(\mu) + \delta_{\ell(\mu), m(\mu)} = \ell(\mu) - m(\mu) \leq \ell.$$

This completes the proof of (ii).  $\square$

It follows from Lemma 2.2 that

$$\delta_{\ell(\mu), m(\mu)} \leq \ell(\pi) - (\ell(\mu) - m(\mu)) \leq m(\mu), \quad (2.1)$$

where  $\ell(\pi) - (\ell(\mu) - m(\mu))$  is the number of multiples of  $t$  in  $\pi$ .

**Lemma 2.3.** *Let  $n$  be a fixed positive integer, and  $n'$  a fixed nonnegative integer. Then the following system of equations*

$$\begin{cases} x + y &= n, \\ s x + (s + 1)y &= n' \end{cases} \quad (2.2)$$

*has exactly one simultaneous solution  $(x, y, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ .*

*Proof.* We readily see that  $y = n' - sn$ . Also, since  $x > 0$  and  $y \geq 0$ , it follows from the first equation that  $0 \leq y < n$ . Hence

$$\frac{n'}{n} - 1 < s \leq \frac{n'}{n},$$

from which it follows that  $s = \lfloor n'/n \rfloor$ . Therefore, there is only one solution  $(x, y, s)$ .  $\square$

We are now ready to determine how many pre-images an overpartition in  $\overline{\mathcal{P}}_t$  has.

**Theorem 2.4.** *Let  $\lambda$  be a nonempty overpartition  $\in \overline{\mathcal{P}}_t$ .*

*(i) If  $\lambda$  has parts equal to  $t$  only, then there are exactly  $2m(\lambda)$  pre-images in  $\overline{\mathcal{G}}_t$  under  $\phi$ . Moreover, of those pre-images, exactly  $m(\lambda)$  pre-images have no overlined parts, and the other  $m(\lambda)$  pre-images have one of the smallest parts overlined.*

*(ii) If  $\lambda$  has parts not equal to  $t$ , then there are exactly  $2m(\lambda) + 1$  pre-images in  $\overline{\mathcal{G}}_t$  under  $\phi$ . Moreover, of those pre-images, exactly  $m(\lambda) + 1$  pre-images have the same number of overlined parts as  $\lambda$  and the other  $m(\lambda)$  pre-images have one more overlined part than  $\lambda$  does.*

*Proof.* Let  $\pi$  be a pre-image of  $\lambda$ . By Lemma 2.2, we know that

$$\ell(\lambda) - m(\lambda) + \delta_{\ell(\lambda), m(\lambda)} \leq \ell(\pi) \leq \ell(\lambda). \quad (2.3)$$

Hence, for any integer  $\ell$  in this range, we want to know how many  $\pi \in \overline{\mathcal{G}}_t$  with  $\ell(\pi) = \ell$  can be pre-images of  $\lambda$ .

In order for  $\pi$  to be a pre-image of  $\lambda$  with  $\ell(\pi) = \ell$ ,  $s(\pi)$  and  $k(\pi)$  must satisfy

$$s(\pi)(\ell - k(\pi)) + (s(\pi) + 1)k(\pi) = m(\lambda). \quad (2.4)$$

By the definition of  $k(\pi)$ , it should be less than  $\ell(\pi)$ , i.e.,  $\ell - k(\pi) > 0$ . Thus, (2.4) is equivalent to that  $(\ell - k(\pi), k(\pi), s(\pi))$  is a solution to (2.2) with  $n = \ell$  and  $n' = m(\lambda)$ , which is unique.

(i) Suppose that  $\lambda$  has parts equal to  $t$  only, i.e.,  $\ell(\lambda) = m(\lambda)$ . By (2.3), there are  $m(\lambda)$  choices for  $\ell$ . For a fixed  $\ell$ ,  $k(\pi)$  and  $s(\pi)$  are uniquely determined as seen above. With these  $(\ell, k(\pi), s(\pi))$ , we can construct  $\pi$ , in which parts are multiples of  $t$  differing by at most  $t$  and there are no overlined parts.

For each  $\pi$ , by having one of the smallest parts overlined, we obtain a different pre-image. Therefore, the total number of pre-images must be equal to  $2m(\lambda)$  as claimed. Also,  $m(\lambda)$  pre-images have no overlined parts and the other  $m(\lambda)$  pre-images have one overlined smallest part.

(ii) Suppose that  $\lambda$  has parts not equal to  $t$ , i.e.,  $\ell(\lambda) > m(\lambda)$ . By (2.3), there are  $(m(\lambda) + 1)$  choices for  $\ell$ . For a fixed  $\ell$ ,  $k(\pi)$  and  $s(\pi)$  are uniquely determined. With these  $(\ell, k(\pi), s(\pi))$ , we can construct  $\pi$ , in which no multiples of  $t$  are overlined.

Note that if  $\ell(\pi) > \ell(\lambda) - m(\lambda)$ , then  $\pi$  must have a multiple of  $t$  as a part. For such  $\pi$ , by having one of the smallest multiples of  $t$  overlined, we obtain a different pre-image.

Therefore, the total number of pre-images must be equal to  $(2m(\lambda) + 1)$  as claimed. Also,  $(m(\lambda) + 1)$  pre-images have the same number of overlined parts as  $\lambda$  and the other  $m(\lambda)$  pre-images have one more overlined part than  $\lambda$  does.  $\square$

Theorem 2.4 yields

$$\sum_{\pi \in \overline{\mathcal{G}}_t} z^{o(\pi)} q^{|\pi|} = \sum_{\lambda \in \overline{\mathcal{P}}_t} ((1 - \delta_{\ell(\lambda), m(\lambda)}) + (1 + z)m(\lambda)) z^{o(\lambda)} q^{|\lambda|}. \quad (2.5)$$

In the following example, we present how to find all the pre-images  $\pi$  of  $\lambda$ .

**Example 2.2.** Let  $t = 3$ .

(i) Let  $\lambda = (3, 3, 3)$ . Since  $\ell(\lambda) = m(\lambda) = 3$ , by Lemma 2.2

$$1 \leq \ell(\pi) \leq 3.$$

By solving (2.4), we have  $(\ell(\pi), k(\pi), s(\pi)) = (1, 0, 3), (2, 1, 1), (3, 0, 1)$ , which yield

$$(9), (\overline{9}),$$

$$(6, 3), (6, \overline{3}),$$

$$(3, 3, 3), (\overline{3}, 3, 3),$$

respectively. There are  $2m(\lambda)$  pre-images.

(ii) Let  $\lambda = (3, 3, 3, \overline{1}, 1)$ . Since  $\ell(\lambda) = 5$  and  $m(\lambda) = 3$ , by Lemma 2.2

$$2 \leq \ell(\pi) \leq 5.$$

By solving (2.4), we have  $(\ell(\pi), k(\pi), s(\pi)) = (2, 1, 1), (3, 0, 1), (4, 3, 0), (5, 3, 0)$ , which yield

$$(7, \overline{4}),$$

$$(\overline{4}, 4, 3), (\overline{4}, 4, \overline{3}),$$

$$(4, 3, 3, \overline{1}), (4, \overline{3}, 3, \overline{1}),$$

$$(3, 3, 3, \overline{1}, 1), (\overline{3}, 3, 3, \overline{1}, 1),$$

respectively. Thus, there are  $2m(\lambda) + 1$  pre-images.

**2.2. Partition Sets  $\overline{\mathcal{P}}_t$  and  $\overline{\mathcal{B}}_t$ .** Let us recall the definition of  $\overline{\mathcal{B}}_t$ , from which it is clear that

$$\sum_{\beta \in \overline{\mathcal{B}}_t} z^{o(\beta)} q^{|\beta|} = (1 + q^t + q^{2t} + \cdots) \left( \frac{(-zq)_t}{(q)_t} - 1 \right)$$

$$= \frac{1}{1 - q^t} \left( \frac{(-zq)_t}{(q)_t} - 1 \right), \quad (2.6)$$

where  $o(\beta)$  denotes the number of overlined parts in  $\beta$ , which is indeed the number of overlined parts in the second subpartition of  $\beta$ .

We now construct a map  $\psi : \overline{\mathcal{B}}_t \rightarrow \overline{\mathcal{P}}_t$  as follows:

- (1) First collect all parts equal to  $t$  in both subpartitions and replace an overlined  $t$  by a non-overlined  $t$ ;
- (2) and then append the remaining parts in the second subpartition to the parts collected in (1).

For example,  $[(3), (3, 3, \overline{1}, 1)]$  and  $[(3), (\overline{3}, 3, \overline{1}, 1)]$  are both mapped to  $(3, 3, 3, \overline{1}, 1)$  under  $\psi$ .

Let  $\lambda \in \overline{\mathcal{P}}_t$ . Suppose that  $\ell(\lambda) = m(\lambda)$ , i.e.,  $\lambda$  has parts equal to  $t$  only. Then, its pre-image  $\beta$  must be a bipartition of this form

$$[(\underbrace{t, \dots, t}_{m(\lambda)-x}), (\underbrace{t, \dots, t}_x)]$$

for some  $x > 0$  with at most one of  $t$ 's in the second subpartition overlined. Thus there are  $2m(\lambda)$  pre-images of  $\lambda$  in  $\overline{\mathcal{B}}_t$  under  $\psi$ . Of those pre-images,  $m(\lambda)$  pre-images have the same number of overlined parts as  $\lambda$ , and the other  $m(\lambda)$  pre-images have one more overlined part than  $\lambda$ .

Suppose that  $\ell(\lambda) > m(\lambda)$ , i.e.,  $\lambda$  has a part not equal to  $t$ . Then, its pre-image  $\pi$  must be a bipartition of this form

$$[(\underbrace{t, \dots, t}_{m(\lambda)-x}), (\underbrace{t, \dots, t}_x, \lambda_{m(\lambda)+1}, \dots)]$$

for some  $x \geq 0$  with at most one of  $t$ 's in the second subpartition overlined. Thus there are  $2m(\lambda) + 1$  pre-images of  $\lambda$  in  $\overline{\mathcal{B}}_t$  under  $\psi$ . Of those pre-images,  $(m(\lambda) + 1)$

pre-images have the same number of overlined parts as  $\lambda$ , and the other  $m(\lambda)$  pre-images have one more overlined part than  $\lambda$ .

Therefore, it follows from the map  $\psi$  that

$$\sum_{\lambda \in \overline{\mathcal{P}}_t} ((1 - \delta_{\ell(\lambda), m(\lambda)}) + (1 + z)m(\lambda)) z^{o(\lambda)} q^{|\lambda|} = \sum_{\beta \in \overline{\mathcal{B}}_t} z^{o(\beta)} q^{|\beta|}. \quad (2.7)$$

By (2.5), (2.6), and (2.7),

$$\sum_{n \geq 1} \sum_{m \geq 0} g_t(m, n) z^m q^n = \sum_{\pi \in \overline{\mathcal{G}}_t} z^{o(\pi)} q^{|\pi|} = \sum_{\beta \in \overline{\mathcal{B}}_t} z^{o(\beta)} q^{|\beta|} = \frac{1}{1 - q^t} \left( \frac{(-zq)_t}{(q)_t} - 1 \right),$$

which completes the proof of Theorem 1.1.

### 3. Final remarks

We remark that, by slightly modifying the first proof of [5, Theorem 2.1], we can also prove Theorem 1.1 analytically.

Let

$${}_{r+1}\phi_s \left( \begin{matrix} a_0, a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right) := \sum_{n \geq 0} \frac{(a_0; q)_n (a_1; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left( (-1)^n q^{\binom{n}{2}} \right)^{s-r} z^n.$$

Then we will need the following identities later.

**Lemma 3.1** (First  $q$ -Chu–Vandermonde Sum [1, Eq. (17.6.2)]). *We have*

$${}_2\phi_1 \left( \begin{matrix} a, q^{-n} \\ c \end{matrix}; q, cq^n/a \right) = \frac{(c/a; q)_n}{(c; q)_n}. \quad (3.1)$$

**Lemma 3.2** ([1, Eq. (17.9.6)]). *We have*

$${}_3\phi_2 \left( \begin{matrix} a, b, c \\ d, e \end{matrix}; q, de/(abc) \right) = \frac{(e/a; q)_\infty (de/(bc); q)_\infty}{(e; q)_\infty (de/(abc); q)_\infty} {}_3\phi_2 \left( \begin{matrix} a, d/b, d/c \\ d, de/(bc) \end{matrix}; q, e/a \right). \quad (3.2)$$

First, note that the generating function for partitions in  $\overline{\mathcal{G}}_t$  with smallest part equal to  $r$  is

$$\frac{(1+z)q^r}{1-q^r} \frac{1+zq^{r+1}}{1-q^{r+1}} \cdots \frac{1+zq^{r+t-1}}{1-q^{r+t-1}} \frac{1}{1-q^{r+t}},$$

in which the coefficient of  $z^m q^n$  counts the number of such overpartitions of  $n$  with exactly  $m$  overlined parts. Hence

$$\begin{aligned} \sum_{n \geq 1} \sum_{m \geq 0} g_t(m, n) z^m q^n &= \sum_{r \geq 1} \frac{(1+z)q^r}{1-q^r} \frac{1+zq^{r+1}}{1-q^{r+1}} \cdots \frac{1+zq^{r+t-1}}{1-q^{r+t-1}} \frac{1}{1-q^{r+t}} \\ &= (1+z) \sum_{r \geq 1} \frac{(q)_{r-1} (-zq)_{r+t-1}}{(q)_{r+t} (-zq)_r} q^r \\ &= (1+z)q \sum_{r \geq 0} \frac{(q)_r (-zq)_{r+t}}{(q)_{r+t+1} (-zq)_{r+1}} q^r \\ &= \frac{(1+z)q(-zq)_t}{(1+zq)(q)_{t+1}} \sum_{r \geq 0} \frac{(q)_r (q)_r (-zq^{t+1})_r}{(q)_r (q^{t+2})_r (-zq^2)_r} q^r \\ &= \frac{(1+z)q(-zq)_t}{(1+zq)(q)_{t+1}} {}_3\phi_2 \left( \begin{matrix} q, q, -zq^{t+1} \\ -zq^2, q^{t+2} \end{matrix}; q, q \right) \end{aligned}$$

$$\begin{aligned}
(\text{by Eq. (3.2)}) &= \frac{(1+z)q(-zq)_t (q^{t+1})_\infty (q^2)_\infty}{(1+zq)(q)_{t+1} (q^{t+2})_\infty (q)_\infty} {}_3\phi_2 \left( \begin{matrix} q, -zq, q^{1-t} \\ -zq^2, q^2 \end{matrix}; q, q^{t+1} \right) \\
&= \frac{(1+z)q(-zq)_t}{(1-q)(1+zq)(q)_t} \sum_{r \geq 0} \frac{(-zq)_r (q^{1-t})_r}{(-zq^2)_r (q^2)_r} q^{r(t+1)} \\
&= -\frac{(-zq)_t}{(1-q^t)(q)_t} \sum_{r \geq 0} \frac{(-z)_{r+1} (q^{-t})_{r+1}}{(-zq)_{r+1} (q)_{r+1}} q^{(r+1)(t+1)} \\
&= -\frac{(-zq)_t}{(1-q^t)(q)_t} \left( {}_2\phi_1 \left( \begin{matrix} -z, q^{-t} \\ -zq \end{matrix}; q, q^{t+1} \right) - 1 \right) \\
(\text{by Eq. (3.1)}) &= -\frac{(-zq)_t}{(1-q^t)(q)_t} \left( \frac{(q)_t}{(-zq)_t} - 1 \right) \\
&= \frac{1}{1-q^t} \left( \frac{(-zq)_t}{(q)_t} - 1 \right).
\end{aligned}$$

## References

1. G. E. Andrews,  $q$ -hypergeometric and related functions, *NIST handbook of mathematical functions*, 419–433, U.S. Dept. Commerce, Washington, DC, 2010.
2. G. E. Andrews, M. Beck, and N. Robbins, Partitions with fixed differences between largest and smallest parts, *Proc. Amer. Math. Soc.* **143** (2015), no. 10, 4283–4289.
3. F. Breuer and B. Kronholm, A polyhedral model of partitions with bounded differences and a bijective proof of a theorem of Andrews, Beck, and Robbins, *Res. Number Theory* **2** (2016), Art. 2, 15 pp.
4. R. Chapman, Partitions with bounded differences between largest and smallest parts, *Australas. J. Combin.* **64** (2016), 376–378.
5. S. Chern, An overpartition analogue of partitions with bounded differences between largest and smallest parts, *Preprint* (2017), arXiv:1702.03462.
6. J. Dousse and B. Kim, An overpartition analogue of the  $q$ -binomial coefficients, *Ramanujan J.* **42** (2017), no. 2, 267–283.

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